The constant U_4^+ may be obtained by matching the Taylor series expansion for the velocity distribution near the wall to empirical expressions for the velocity further from the wall [1, 3]. Unfortunately, no universal relation appears to exist for the turbulent temperature distribution as yet. However, equation (16) suggests $(\varepsilon_H/\varepsilon_M)$ is at most a function of the Prandtl number for the simplest case [5].

CONCLUSIONS

It is observed that the eddy diffusivity of heat varies with the cubic power of the wall distance with or without inclusion of transpiration and dissipation effects. This disagrees with the assumption of Deissler that the eddy diffusivity varies as the square of the distance at low Prandtl numbers and with the fourth power at high Prandtl numbers [2]. However, it is in agreement with the very successful analysis presented by Lin *et al.* that the diffusivity goes as the third power near the wall. The analytical formulations of Lin *et al.* correlated heat and mass transport data over a Prandtl or Schmidt number range from 0·54 to 3200. This note provides analytical justification for their empirical usage of an eddy diffusivity coefficient proportional to the third power of the wall distance.

REFERENCES

1. C. L. Tien and D. T. Wasan, Law of the wall in turbulent channel flow, *Physics Fluids* **6**(1), 144–145 (1963).
2. C. L. Tien, A note on distributions of temperature and eddy diffusivities for heat in turbulent flow near a wall, *Z. Angew. Math. Phys.* **15**(1), 63–66 (1964).
3. R. N. Meroney, Velocity and shear distributions in a transpired boundary layer, Proceedings of 10th Midwestern Mechanics Conference, Colorado State University (1967).
4. J. Laufer, The structure of turbulence for a fully developed pipe flow, NACA TR 1174 (1954).
5. W. M. Rösenhow and H. W. Choi, *Heat, Mass and Momentum Transfer*, p. 185. Prentice Hall, New Jersey (1961).
6. C. S. Lin, R. W. Moulton and G. L. Putnam, Mass transfer between solid wall and fluid streams, *Ind. Engng Chem.* **45**(3), 636–640 (1953).

Int. J. Heat Mass Transfer. Vol. 11, pp. 1408–1412. Pergamon Press 1968. Printed in Great Britain

HEAT TRANSFER IN THERMAL ENTRANCE REGION OF COCURRENT FLOW HEAT EXCHANGERS WITH FULLY DEVELOPED LAMINAR FLOW

WILLIAM N. GILL, EARNEST W. PORTA and RICHARD J. NUNGE

Department of Chemical Engineering, Clarkson College of Technology, Potsdam, New York, U.S.A.

(*Received* 3 *October* 1967 *and in revised form* 1 *April* 1968)

NOMENCLATURE

A,	dimensionless velocity gradient at the wall between streams, defined below equation (6);
a_k,	coefficients of expansion in the wall temperature distribution, defined below equation (15);
C_p,	heat capacity;
D_h,	equivalent diameter of the annulus;
H,	capacity ratio, $\frac{w_2 C_{p_2}}{w_1 C_{p_1}}$;
K,	conductivity ratio, k_1/k_2;
k,	thermal conductivity;
Nu,	Nusselt number, defined in equations (16) and (17);
Pe,	Péclet number, $u_B D_h/\alpha$;
r,	radial coordinate;
r_0, r_1, r_2, r_3,	radii defined on Fig. 1;
r^*,	dimensionless radius ratio, defined on Fig. 1;
T,	temperature;
u,	axial velocity;
w,	mass flow rate;
x,	axial coordinate;
$\bar{x}$,	dimensionless axial coordinate, $4(x/D_{h_1})/Pe_1$;
$\bar{x}_2$,	$4(x/D_{h_2})/Pe_2$;
$\bar{y}$,	dimensionless radial coordinate, defined below equation (6).

Greek symbols

α,	thermal diffusivity;
η,	similarity variable, defined below equation (6);
θ,	dimensionless temperature, $\frac{T_i - T_{e_1}}{T_w - T_{e_1}}$;

θ_{ik}, functions tabulated in [1];
$\bar{\lambda}$, parameter in the Duhamel theorem;
ξ, dimensionless axial distance, defined below equation (6);
ϕ, dimensionless temperature, $\frac{T_i - T_{e_1}}{T_{e_2} - T_{e_1}}$;
ω^2, $\frac{(1 + r_1^*)(1 - r_2^*)}{(1 - r_1^*)(1 + r_2^*)} KH$.

Subscripts
B, bulk;
e, entrance;
w, wall;
0, overall;
1, stream 1;
2, stream 2.

INTRODUCTION

RECENTLY Worsøe-Schmidt [1] presented an extension of the classical Lévêque problem by using a series type expansion first suggested by Mercer [2], and later generalized to non-Newtonian flow, cases with mass transfer and to annuli by Gill *et al.* [3–5] It is our purpose here to demonstrate how the solutions for single-stream forced convection problems can be used to construct the temperature distribution in the thermal entrance region of multistream cocurrent flow heat exchangers and to present multistream results for those cases which can be handled using the numerical results of Worsøe-Schmidt. These solutions may be matched with generalized orthogonal function expansions [6–9] which are most convenient to use at larger distances from the inlet, and thereby one can analyze the entire flow field. References [6, 7] consider the more complicated cases which include turbulent flow and axial conduction in the fluid streams respectively.

THEORETICAL DEVELOPMENT

The geometry of the exchanger is shown schematically in Fig. 1. Ignoring axial conduction and viscous dissipation and assuming that the fluid properties are independent of the temperature, each stream in the exchanger is described by

$$u(r)\frac{\partial T_i}{\partial x} = \frac{\alpha_i}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_i}{\partial r}\right); \qquad i = 1, 2 \tag{1}$$

and the boundary conditions are

$$T_i(0, r) = T_{e_i} \tag{2}$$

$$\frac{\partial T_1}{\partial r}(x, r_1) = \frac{\partial T_2}{\partial r}(x, r_0) = 0 \tag{3}$$

$$T_1(x, r_2) - T_2(x, r_3) = -\frac{k_1}{k_w} r_2 \ln(r_3/r_2) \frac{\partial T_1}{\partial r}(x, r_2) \tag{4}$$

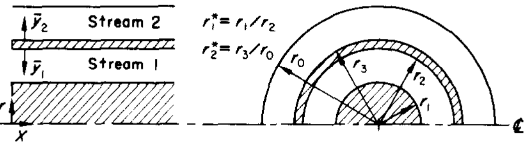

FIG. 1. Schematic diagram of a cocurrent flow double-annulus heat exchanger.

$$\frac{k_1 r_2}{k_2 r_3}\frac{\partial T_1}{\partial r}(x, r_2) = \frac{\partial T_2}{\partial r}(x, r_3). \tag{5}$$

The constant multiplying the temperature gradient on the right-hand side of equation (4) is a measure of the resistance of the wall between streams. Earlier investigations [8] have indicated that unless the wall resistance is very large, it has only small effects on the results. Hence for simplicity consider that the fluid temperatures at either side of the wall between streams are equal, that is, that the right-hand side of equation (4) is zero.

To solve equations (1–5) consider each stream separately, and find the single-stream temperature distributions using equations (1–3) and either a uniform wall temperature or a uniform wall flux at the boundary between streams. Since previous studies [6–9] for regions away from the entrance have shown that for cocurrent flow, the system more closely follows the uniform wall temperature solution for single stream flow, we shall employ that condition here. Then for the thermal entry region the temperature distribution given by Worsøe-Schmidt is

$$\theta_i = \frac{T_i - T_{e_i}}{T_w - T_{e_i}} = \sum_{k=0}^{\infty} \xi_i^k \theta_{ik}(\eta_i); \; i = 1, 2 \tag{6}$$

where

$$\xi_i = \left(\frac{36x}{Pe_i A_i D_{h_i}}\right)^{\frac{1}{3}}$$

$$\eta_i = \bar{y}_i/\xi_i$$

$$A_1 = \frac{2(r_1^* - 1)}{1 + r_1^{*2} + (1 - r_1^{*2})/\ln r_1^*}\left(\frac{1 - r_1^{*2}}{-\ln r_1^*} - 2\right),$$

$$A_2 = \frac{2(l - r_2^*)}{r_2^*[1 + r_2^{*2} + (1 - r_2^{*2})/\ln r_2^*]}\left(\frac{1 - r_2^{*2}}{-\ln r_2^*} - 2r_2^{*2}\right)$$

and

$$\bar{y}_1 = \frac{1 - r/r_2}{1 - r_1/r_2} = \frac{1 - r/r_2}{1 - r_1^*}$$

$$\bar{y}_2 = \frac{r/r_0 - r_3/r_0}{1 - r_3/r_0} = \frac{r/r_0 - r_2^*}{1 - r_2^*}.$$

This result can be generalized to a wall temperature which is an arbitrary function of x by employing the Duhamel theorem. In the form of the Duhamel theorem given by Bartells and Churchill [10] the single-stream temperature distributions for an arbitrary wall temperature distribution, which is the same for both streams in view of the assumption relating to equation (4), are

$$T_i = \frac{\partial}{\partial x}\int_0^x \{[T_w(\lambda) - T_{e_i}]\,\theta_i(\lambda, x-\lambda, \bar{y}_i) + T_{e_i}\}\,d\lambda. \quad (7)$$

This solution satisfies all the conditions of the original problem except equation (5). In terms of the dimensionless coordinates $\bar{y}_i$, equation (5) becomes

$$-K\frac{(1-r_2^*)}{r_2^*(1-r_1^*)}\frac{\partial T_1}{\partial \bar{y}_1}(x,0) = \frac{\partial T_2}{\partial \bar{y}_2}(x,0). \quad (8)$$

Substituting equation (7) in equation (8) and evaluating the derivatives of θ_i with respect to $\bar{y}_i$ at $\bar{y}_i = 0$, we have from equation (6)

$$\left.\frac{\partial\theta_i}{\partial\bar{y}_i}\right|_{y_i=0} = \left.\frac{\partial\theta_i}{\partial\eta_i}\right|_{\eta_i=0}\frac{\partial\eta_i}{\partial\bar{y}_i} = \sum_{k=0}^{\infty}\xi_i^{k-1}\theta'_{ik}(0)$$

$$= \sum_{k=0}^{\infty}\left(\frac{36}{Pe_iA_iD_{h_i}}\right)^{(k-1)/3} x^{(k-1)/3}\theta'_{ik}(0). \quad (9)$$

Hence equation (8) becomes

$$\frac{-K(1-r_2^*)}{r_2^*(1-r_1^*)}\frac{\partial}{\partial\bar{x}}\int_0^{\bar{x}}\phi_w(\bar{\lambda})\sum_{k=0}^{\infty}\left(\frac{9}{A_1}\right)^{(k-1)/3}\theta'_{ik}(0)(\bar{x}-\bar{\lambda})^{(k-1)/3}d\bar{\lambda}$$

$$= \frac{\partial}{\partial\bar{x}}\int_0^{\bar{x}}\{\phi_w(\bar{\lambda})-1\}\sum_{k=0}^{\infty}\left(\frac{9}{\omega^2A_2}\right)^{(k-1)/3}\theta'_{2k}(0) \times (\bar{x}-\bar{\lambda})^{(k-1)/3}\,d\bar{\lambda} \quad (10)$$

The capacity ratio, H, is a parameter commonly used in describing heat exchanger performance. Equation (10) can be rearranged to

$$\sum_{k=0}^{\infty}\left[\left(\frac{9}{\omega^2A_2}\right)^{(k-1)/3}\theta'_{2k}(0) + K\frac{(1-r_2^*)}{r_2^*(1-r_1^*)}\left(\frac{9}{A_1}\right)^{(k-1)/3}\theta'_{1k}(0)\right] \times\frac{\partial}{\partial\bar{x}}\int_0^{x}\phi_w(\bar{\lambda})(\bar{x}-\bar{\lambda})^{(k-1)/3}d\bar{\lambda}$$

$$= \sum_{k=0}^{\infty}\left(\frac{9}{\omega^2A_2}\right)^{(k-1)/3}\theta'_{2k}(0)\frac{\partial}{\partial\bar{x}}\int_0^{\bar{x}}(\bar{x}-\bar{\lambda})^{(k-1)/3}d\bar{\lambda}. \quad (11)$$

The right-hand side of equation (11) can be simplified by noting that

$$\frac{\partial}{\partial\bar{x}}\int_0^{\bar{x}}(\bar{x}-\bar{\lambda})^{(k-1)/3}d\bar{\lambda} = \bar{x}^{(k-1)/3}, \quad k = 0,1,2,\ldots \quad (12)$$

Equation (11) is a Volterra integral equation of the first kind which can be solved for $\phi_w(\bar{x})$ by employing the Laplace transform and recognizing that the left-hand side is in the form of the convolution integral. That is,

$$L\left\{\frac{\partial}{\partial\bar{x}}\int_0^{\bar{x}}\phi_w(\bar{\lambda})(\bar{x}-\bar{\lambda})^{(k-1)/3}d\bar{\lambda}\right\} = p\bar{\phi}_w(p)\,L[\bar{x})^{(k-1)/3}]$$

$$= p\bar{\phi}_w(p)\,\Gamma\left(\frac{k+2}{3}\right)p^{-(k+2)/3}. \quad (13)$$

Thus taking the Laplace transform of both sides of equation (11) and rearranging yields

$$\bar{\phi}_w(p) = \frac{\sum_{k=0}^{\infty}\left(\frac{9}{\omega^2A_2}\right)^{(k-1)/3}\theta'_{2k}(0)\,\Gamma\left(\frac{k+2}{3}\right)p^{-(k+2)/3}}{\sum_{k=0}^{\infty}\left[\left(\frac{9}{\omega^2A_2}\right)^{(k-1)/3}\theta'_{2k}(0) + K\frac{(1-r_2^*)}{r_2^*(1-r_1^*)}\left(\frac{9}{A_1}\right)^{(k-1)/3}\theta'_{1k}(0)\right]\Gamma\left(\frac{k+2}{3}\right)p^{-(k-1)/3}}$$

$$= \sum_{k=0}^{\infty}b_kp^{-(k+2)/3}\Big/\sum_{k=0}^{\infty}c_kp^{-(k-1)/3}. \quad (14)$$

where

$$\phi_w = \frac{T_w - T_{e_1}}{T_{e_2} - T_{e_1}}$$

$$\bar{x} = \frac{4x}{Pe_1D_{h_1}}$$

$$\omega^2 = \frac{Pe_2D_{h_2}}{Pe_1D_{h_1}} = \frac{(1+r_1^*)(1-r_2^*)}{(1-r_1^*)(1+r_2^*)}KH.$$

Term by term inversion of the quotient resulting from dividing out the right-hand side of equation (14) gives

$$\phi_w(\bar{x}) = \sum_{k=0}^{\infty}a_k\bar{x}^{k/3} \quad (15)$$

where

$$a_0 = b_0/c_0$$

$$a_k = \left(b_k - \sum_{n=0}^{k-1}a_nc_{k-n}\right)\Big/\left[c_0\,\Gamma\left(\frac{k+3}{3}\right)\right], \quad k \geq 1.$$

The results of Worsøe-Schmidt provide seven terms for the summation in equation (15).

RESULTS

Once the wall temperature is known, it is an easy task to calculate the Nusselt numbers which describe the characteristics of the entrance region. The individual stream Nusselt numbers are defined as

$$Nu_i = \frac{-2\,\partial\phi_i/\partial\bar{y}_i\,(\bar{x},0)}{\phi_{iB}(\bar{x}) - \phi_w(\bar{x})} \tag{16}$$

and the overall Nusselt number based on the properties of stream 1 is given by

$$Nu_0 = \frac{-2\,\partial\phi_1/\partial\bar{y}_1\,(\bar{x},0)}{\phi_{1B}(\bar{x}) - \phi_{2B}(\bar{x})} \tag{17}$$

where the subscript B denotes the bulk quantity. Using equation (7) and an overall heat balance it can easily be demonstrated that

$$Nu_1 = \frac{-2\dfrac{\partial}{\partial\bar{x}}\displaystyle\int_0^{\bar{x}} \phi_w(\bar{\lambda}) \sum_{k=0}^{\infty} \left(\frac{9}{A_1}\right)^{(k-1)/3} (\bar{x}-\bar{\lambda})^{(k-1)/3}\,\theta'_{ik}(0)\,\mathrm{d}\bar{\lambda}}{\dfrac{8}{(1+r_1^*)}\displaystyle\int_0^{\bar{x}} \frac{\partial\phi_1}{\partial\bar{y}_1}(\bar{x},0)\,\mathrm{d}\bar{x} - \phi_w(\bar{x})} \tag{18}$$

Similar expressions can be developed for Nu_2 and Nu_0.

Local Nusselt numbers, bulk temperatures and wall temperatures for double-pipe, cocurrent flow heat exchangers are available in [8]. The values shown in Table 1 are typical of the agreement obtained between the similarity approach used here and the eigenvalue solution given in [8]. The entrance region results are in excellent agreement up to dimensionless lengths of $\bar{x} = 0{\cdot}2$. Nusselt number and temperature distributions are extended in Table 1 to smaller dimensionless lengths using the similarity approach.

Several combinations of the parameters H, K, r_1^* and r_2^* were used to calculate the local Nusselt numbers. Plots of Nu_1 vs. $\bar{x}$ at constant r_1^* revealed only small variations of Nu_1 with H, K and r_2^* over wide ranges of these parameters. A similar behavior was observed for Nu_2 plotted vs. $\bar{x}_2 = \bar{x}/\omega^2$ at constant r_2^*. H and K were varied between 0·1 and 20 and r_1^* and r_2^* took on the values investigated by Worsøe-Schmidt. The individual stream Nusselt numbers were compared with the single-stream Nusselt numbers given in [1] for a step change in the wall temperature (UWT) and for a step change in the heat flux (UWF). As shown on Fig. 2, the multistream Nusselt numbers correlated well with the UWT result. Results for several other sets of parameters applicable to but not shown on Fig. 2 were investigated and correlated equally as well as the information shown. The additional sets of parameters included: $H = 1{\cdot}0$, $K = 1{\cdot}0$; $H = 1{\cdot}0$, $K = 0{\cdot}1$, and $H = 0{\cdot}1$, $K = 1{\cdot}0$. This leads to the conclusion that by using the single-stream UWT Nusselt numbers as a function of $\bar{x}$ and $\bar{x}_2$ in combination with the additivity of resistance concept,

$$\frac{1}{Nu_0(\bar{x})} = \frac{1}{Nu_1(\bar{x})} + \frac{K\dfrac{(1-r_2^*)}{(1-r_1^*)\,r_2^*}}{Nu_2(\bar{x}_2)}$$

the local overall Nusselt number in the entrance region can be successfully predicted for cocurrent flow. Over the range of parameters investigated here, the maximum error in predicting the overall Nusselt number was 6 per cent. The results are generally more accurate at the smallest axial distances and in most cases the error was much less than 6 per cent. All the numerical results are tabulated in [11].

Despite the agreement of the Nusselt numbers with the single-stream UWT results, the wall temperature in the

Table 1. A comparison of the entrance region results obtained by the similarity method and the eigenvalue method [8] for a cocurrent flow, double-pipe heat exchanger with $H = 2{\cdot}25$, $K = 2{\cdot}0$, $r_1^* = 0$, and $r_2^* = 0{\cdot}5$

$\bar{x}$	ϕ_{1B}	ϕ_{1B}†	ϕ_w	ϕ_w†	Nu_1	Nu_1†	Nu_2	Nu_2†	Nu_0	Nu_0†
1×10^{-4}	0·00224		0·416		35·96		51·00		14·92	
2×10^{-4}	0·00353		0·418		28·41		40·58		11·84	
1×10^{-3}	a·0103		0·426		16·42		23·96		6·93	
2×10^{-3}	0·0163		0·431		12·98		19·15		5·51	
1×10^{-2}	0·0470		0·449		7·64		11·56		3·29	
2×10^{-2}	0·0737	0·0737	0·460	0·460	6·17	6·17	9·42	9·42	2·67	2·67
4×10^{-2}	0·115	0·115	0·476	0·476	5·09	5·09	7·78	7·78	2·21	2·21
1×10^{-1}	0·205	0·205	0·506	0·506	4·20	4·20	6·28	6·28	1·80	1·80
2×10^{-1}	0·311	0·311	0·540	0·541	3·98	3·93	5·65	5·63	1·65	1·64
4×10^{-1}	0·457	0·453	0·591	0·594	4·34	3·92	5·69	5·38	1·72	1·60
6×10^{-1}	0·553	0·541	0·636	0·629	4·66	3·94	6·59	5·35	1·93	1·59

† Eigenvalue solution.

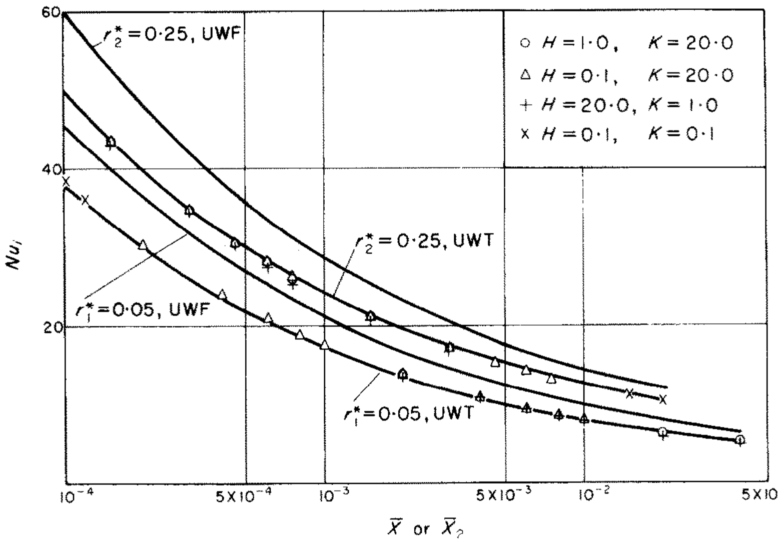

FIG. 2. A comparison of individual stream Nusselt numbers for cocurrent flow calculated from the similarity approach for several sets of the parameters H and K with the single-stream results for UWF and UWT [1] at $r_1^* = 0{\cdot}05$ and $r_2^* = 0{\cdot}25$.

cocurrent flow exchanger can exhibit significant variations. Over the range of $\bar{x}$ reported here the dimensionless wall temperature changed by a maximum of 27 per cent for $H = 0{\cdot}1$, $K = 20$, $r_2^* = 0{\cdot}25$ and $r_1^* = 0{\cdot}05$. The majority of the wall temperatures showed variations of about 15 per cent.

CONCLUSION

Although the method developed here was applied only to cocurrent flow systems, it is believed that a similar approach involving the Duhamel theorem can be used for other multistream problems such as countercurrent flow systems and we are seeking such extensions.

ACKNOWLEDGEMENT

Computations were performed at the Clarkson College Computing Center and the work was supported by the Office of Naval Research under Contract 5053(01).

REFERENCES

1. P. M. WORSØE-SCHMIDT, Heat transfer in the thermal entrance region of circular tubes and annular passages with fully developed laminar flow, *Int. J. Heat Mass Transfer* **10**, 541–555 (1967).
2. A. M. MERCER, The growth of the thermal boundary layer at the inlet to a circular tube, *Appl. Scient. Res.* **A9**, 450–456 (1960).
3. W. N. GILL, The thermal entrance region in laminar power law flows, *Appl. Scient. Res.* **A11**, 10–12 (1962).
4. W. N. GILL, E. DEL CASAL and D. W. ZEH, Free and forced convection in conduits with asymmetric mass transfer, *A.I.Ch.E. Jl* **12**, 266–271 (1966).
5. W. N. GILL, D. W. ZEH and C. TIEN, Reverse osmosis in annuli, *A.I.Ch.E. Jl* **12**, 1141–1146 (1966).
6. J. A. BLANCO and W. N. GILL, Analysis of multistream turbulent forced convection systems, *Chem. Engng Prog. Symp. Ser.* 77 **63**, 66 (1967).
7. R. J. NUNGE, E. W. PORTA and W. N. GILL, Axial conduction in the fluid streams of multistream heat exchangers, *Chem. Engng Prog. Sym. Ser.* 77, **63**, 80 (1967).
8. R. J. NUNGE and W. N. GILL, An analytical study of laminar counterflow double-pipe heat exchangers, *A.I.Ch.E. Jl* **12**, 279–289 (1966).
9. R. P. STEIN, *Adv. Heat Transf.* **3** (1966).
10. C. F. BARTELLS and R. V. CHURCHILL, Resolution of boundary problems by the use of a generalized convolution, *Bull. Am. Math. Soc.* **48**, 276 (1942).
11. E. W. PORTA, Ph.D. Dissertation, Clarkson College, in progress (1967).